

A novel nontrivial bosonic representation of spin operators in the large N limit at high temperatures

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Abstract

We report on a nontrivial bosonization scheme for spin operators. It is shown that in the large N limit, at infinite temperature, the operators $\sum_{k=1}^N \hat{s}_{k\pm}/\sqrt{N}$ behave like the creation and annihilation operators, a^\dagger and a , corresponding to a harmonic oscillator in thermal equilibrium, whose temperature and frequency are related by $\hbar\omega/k_B T = \ln 3$. The z component is found to be equivalent to the position variable of another harmonic oscillator occupying its ground Gaussian state at zero temperature. The obtained results are applied to the Heisenberg XY Hamiltonian at finite temperature.

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I. INTRODUCTION

The interest in bosonic representations of spin operators arises from the necessity of finding alternative practical tools to the diagrammatic technique for spins which turns out to be quite complicated [1]. By using bosonic operators, one can, for instance, evaluate the functional integrals and afterwards calculate the different contributions diagrammatically [2–4]. Moreover, spin-wave theory is based on the possibility of transforming spin operators into new bosonic operators, which are much more easier to deal with [5]. This theory has known great success in the study of magnetic properties of many materials at low temperatures; it led, in particular, to the notion of magnon, the elementary collective excitation of spins in the crystal lattice. Many important spin Hamiltonians, such as the Heisenberg model Hamiltonian, can be exactly diagonalized at low excitations with respect to the bosonic degrees of freedom using certain transformations.

Among the most used transformations in spin-wave theory we cite the Dyson [6] and the Holstein-Primakoff transformations. The latter maps spin- $\frac{1}{2}$ operators as follows [7]:

$$\hat{s}_+ = a^\dagger \sqrt{1 - a^\dagger a}, \quad (1)$$

$$\hat{s}_- = \sqrt{1 - a^\dagger a} a, \quad (2)$$

$$\hat{s}_z = -\frac{1}{2} + a^\dagger a. \quad (3)$$

Here and throughout the text, unless otherwise stated, we take $\hbar=1$. For a set of N spins, the scaled total raising operator is thus given by

$$\frac{\hat{S}_+}{\sqrt{N}} = \frac{1}{\sqrt{N}} \sum_{k=1}^N a_k^\dagger \sqrt{1 - a_k^\dagger a_k}. \quad (4)$$

At low temperatures and low excitations, i.e. when $\langle a_k^\dagger a_k \rangle \ll 1$, we may, as a first approximation, write

$$\frac{\hat{S}_+}{\sqrt{N}} = \frac{1}{\sqrt{N}} \sum_{k=1}^N a_k^\dagger. \quad (5)$$

For large N , the Fourier transforms of the bosonic operators a_k and a_k^\dagger are given by

$$\alpha_\ell = \frac{1}{\sqrt{N}} \sum_k e^{-ik\ell} a_k, \quad \alpha_\ell^\dagger = \frac{1}{\sqrt{N}} \sum_k e^{ik\ell} a_k^\dagger. \quad (6)$$

These are also bosonic operators, since they verify the commutation relation $[\alpha_\ell, \alpha_{\ell'}^\dagger] = \delta_{\ell\ell'}$. We can thus conclude that at low excitations (temperatures), and large N ,

$$\frac{\hat{S}_+}{\sqrt{N}} = \alpha_0^\dagger, \quad \frac{\hat{S}_-}{\sqrt{N}} = \alpha_0 \quad (7)$$

which is indeed a beautiful result. However, at higher excitations, the square root in equations (1)-(3) should be expanded in powers of $a^\dagger a$, depending on the order of the adopted approximation. This gives, for instance,

$$\frac{\hat{S}_+}{\sqrt{N}} = \frac{1}{\sqrt{N}} \sum_{k=1}^N a_k^\dagger \left(1 - \frac{a_k^\dagger a_k}{2} - \frac{(a_k^\dagger a_k)^2}{8} - \frac{(a_k^\dagger a_k)^3}{16} - \dots \right). \quad (8)$$

In this paper we address the problem of finding a suitable bosonic representation similar to (7) that is valid at high temperatures. As we shall see, the cost we have to pay for preserving the above form of the transformation resides in the introduction of effective temperatures that are generally different from the actual ones, together with a deformation of the functions of interest.

The paper is organized as follows. In Section II we prove some fundamental results in connection with the addition of spin operators. The focus is on the trace properties of the suitably scaled operators to ensure good statistical behaviour in the large N limit. Our main result is presented in Section III where we explicitly establish the exact form of the mapping between the spin and bosonic operators. The paper is ended with a brief discussion.

II. PRELIMINARY RESULTS

Let us begin by proving some results about the behaviour of the spin operators $\hat{S}_\alpha/\sqrt{N} = \sum_{i=1}^N \hat{s}_{i\alpha}/\sqrt{N}$ ($\alpha \equiv x, y, z$) as $N \rightarrow \infty$ [8–12]. By the multinomial theorem we have that

$$\text{tr}(\hat{S}_\alpha)^\ell = \sum_{r_1 r_2 \dots r_N} \frac{\ell!}{r_1! r_2! \dots r_N!} \text{tr} \hat{s}_{1\alpha}^{r_1} \otimes \hat{s}_{2\alpha}^{r_2} \otimes \dots \otimes \hat{s}_{N\alpha}^{r_N} \delta\left(\ell - \sum_k r_k\right). \quad (9)$$

Note that we are using the Kronecker delta symbol. We have to distinguish between two possible cases, depending on whether ℓ is odd or even. In the former case, we see that there should always be an operator, in each term of the right-hand side of the above equation, appearing with an odd power, say, $2n+1$ where $n \in \mathbb{Z}_+$. But since $\text{tr} \hat{s}_\alpha^{2n+1} = 0$ for all n , we conclude that

$$\text{tr} \left(\frac{\hat{S}_\alpha}{\sqrt{N}} \right)^{2\ell+1} = 0, \quad \forall \ell \in \mathbb{Z}_+. \quad (10)$$

When the power of the operator is even,

$$\text{tr}(\hat{S}_\alpha)^{2\ell} = \sum_{r_1 r_2 \dots r_N} \frac{2\ell!}{r_1! r_2! \dots r_N!} \prod_{k=1}^N \text{tr} \hat{S}_{k\alpha}^{r_k} \delta\left(2\ell - \sum_k r_k\right). \quad (11)$$

One can easily see that the sum in the right-hand side of equation (11) yields a polynomial in N of degree ℓ , the main contribution of which comes from the term in which all the operators $\hat{S}_{k\alpha}$ contribute equally to the product. This happens when all the operators appear with the same power; all the other possibilities yield polynomials of degree less than ℓ . But due to the restriction imposed on the sum of the powers of the operators, we conclude that:

$$\text{tr}(\hat{S}_\alpha)^{2\ell} = \underbrace{\frac{2\ell!}{2! \times 2! \dots 2!}}_{\ell \text{ terms}} \sum_{\Pi_\ell} \left(\prod_{k=1}^{\ell} \text{tr} s_{\alpha k}^2 \prod_{k=\ell+1}^N \text{tr} \mathbb{I}_2 \right)_{\Pi_\ell[1,2,\dots,N]} + Q_{\ell-1}(N), \quad (12)$$

where the products are evaluated for all the possible partitions $\Pi_\ell[1, 2, \dots, N]$ of N elements into subsets of ℓ elements, and $Q_{\ell-1}(N)$ is a polynomial in N of degree at most equal to $\ell-1$. Next we remark that for spin-1/2 operators, $\hat{s}_\alpha^2 = \frac{\mathbb{I}_2}{4}$, meaning that $\text{tr} \hat{s}_\alpha^2 = \frac{1}{2}$, independent of the partition Π_ℓ . Hence we can write

$$\begin{aligned} \text{tr}(\hat{S}_\alpha)^{2\ell} &= \frac{2\ell!}{2^\ell} \frac{N(N-1)(N-2)\dots(N-\ell+1)}{\ell!} (\text{tr } s_\alpha^2)^\ell (\text{tr } \mathbb{I}_2)^{N-\ell} + Q_{\ell-1}(N) \\ &= 2^N N^\ell \left[\frac{(2\ell)!}{2^{3\ell} \ell!} \prod_{k=1}^{\ell-1} \left(1 - \frac{k}{N}\right) + O\left(\frac{1}{N}\right) \right]. \end{aligned} \quad (13)$$

It follows that

$$\lim_{N \rightarrow \infty} 2^{-N} \text{tr}(\hat{S}_\alpha / \sqrt{N})^{2\ell} = \frac{(2\ell)!}{2^{3\ell} \ell!}, \quad \alpha \equiv x, y, z. \quad (14)$$

Now we shall prove that as $N \rightarrow \infty$ the operators $\hat{S}_\alpha / \sqrt{N}$ and \hat{S}_β / \sqrt{N} become uncorrelated (i.e. independent) with respect to the tracial state when $\alpha \neq \beta$. We have

$$\begin{aligned} \text{tr} \left\{ (\hat{S}_\alpha)^m (\hat{S}_\beta)^\ell \right\} &= \sum_{k_1, k_2, \dots, k_N, r_1, r_2, \dots, r_N} \frac{m! \ell!}{r_1! r_2! \dots r_N! k_1! k_2! \dots k_N!} \\ &\times \text{tr} \hat{S}_{1\alpha}^{r_1} \otimes \hat{S}_{2\alpha}^{r_2} \otimes \dots \otimes \hat{S}_{N\alpha}^{r_N} \cdot \hat{S}_{1\beta}^{k_1} \otimes \hat{S}_{2\beta}^{k_2} \otimes \dots \otimes \hat{S}_{N\beta}^{k_N} \\ &\times \delta\left(m - \sum_i r_i\right) \delta\left(\ell - \sum_i k_i\right) \\ &= \sum_{k_1, k_2, \dots, k_N, r_1, r_2, \dots, r_N} \frac{m! \ell!}{r_1! r_2! \dots r_N! k_1! k_2! \dots k_N!} \\ &\times \text{tr}(\hat{S}_{1\alpha}^{r_1} \cdot \hat{S}_{1\beta}^{k_1}) \otimes (\hat{S}_{2\alpha}^{r_2} \cdot \hat{S}_{2\beta}^{k_2}) \otimes \dots \otimes (\hat{S}_{N\alpha}^{r_N} \cdot \hat{S}_{N\beta}^{k_N}) \\ &\times \delta\left(m - \sum_i r_i\right) \delta\left(\ell - \sum_i k_i\right). \end{aligned} \quad (15)$$

Once again, if one of the powers is odd, then in each term in the right-hand side of the latter equation there should be an operator \hat{s}_j appearing with an odd power; this means that its trace is zero, and hence the overall trace vanishes as well. This gives

$$\text{tr}(\hat{S}_\alpha^{2\ell+1}\hat{S}_\beta^m) = \text{tr}(\hat{S}_\alpha)^{2\ell+1}\text{tr}(\hat{S}_\beta)^m = 0, \quad \forall m \in \mathbb{Z}_+. \quad (16)$$

Consider now the case in which the powers of the total spin operators are even and make the substitution $m \rightarrow 2m$, $\ell \rightarrow 2\ell$ in equation (15). It follows that the right-hand side of the latter equation yields a polynomial in N of degree $\ell + m$; we shall assume that $\ell > m$. In this instance, the leading contribution to the sum comes from the terms in which all the spins equally contribute to the trace, which occurs only if all their powers are equal to 2, taking, obviously, into account all the possible partitions of N elements into disjoint subsets of ℓ and m elements. Since we have assumed that $\ell > m$, we should have a contribution to the trace of m operators in the form $\hat{s}_{j\alpha}^2 \hat{s}_{j\beta}^2$, and $\ell - m$ in the form $\hat{s}_{j\beta}^2$; the remaining $N - \ell$ traces come from the identity matrix. More precisely:

$$\begin{aligned} \text{tr}\left\{(\hat{S}_\alpha)^{2m}(\hat{S}_\beta)^{2\ell}\right\} &= \frac{(2m)!(2\ell)!}{\underbrace{2!2!\dots 2!}_{\ell \text{ terms}} \underbrace{2!2!\dots 2!}_{m \text{ terms}}} \sum_{\Pi_\ell \Pi_m} \left(\prod_{j=1}^m \text{tr}(\hat{s}_{j\alpha}^2 \hat{s}_{j\beta}^2) \prod_{j=1}^{\ell-m} \text{tr} \hat{s}_{j\beta}^2 \prod_{j=1}^{N-\ell} \text{tr} \mathbb{I}_2 \right)_{\Pi_\ell, \Pi_m} \\ &+ P_{\ell+m-1}(N), \end{aligned} \quad (17)$$

where $P_{\ell+m-1}(N)$ is a polynomial in N of degree at most equal to $\ell + m - 1$. Taking into account the fundamental properties of spin- $\frac{1}{2}$ operators, we find that

$$\begin{aligned} \text{tr}\left\{(\hat{S}_\alpha)^{2m}(\hat{S}_\beta)^{2\ell}\right\} &= \frac{(2m)!(2\ell)!}{2^{\ell+m}} \times \frac{N(N-1)(N-2)\dots(N-m+1)}{m!} \\ &\times \frac{N(N-1)(N-2)\dots N(N-\ell+1)}{\ell!} \times \frac{2^{N-\ell}}{8^m 2^{\ell-m}} + P_{\ell+m-1}(N) \\ &= N^{\ell+m} \left[\frac{2^N (2m)!(2\ell)!}{2^{3\ell} 2^{3m} \ell! m!} \prod_{k=1}^{\ell-1} \left(1 - \frac{k}{N}\right) \prod_{k'=1}^{m-1} \left(1 - \frac{k'}{N}\right) + O\left(\frac{1}{N}\right) \right]. \end{aligned} \quad (18)$$

As a direct result of equation (18) we deduce that

$$\lim_{N \rightarrow \infty} 2^{-N} \text{tr} \left\{ (\hat{S}_\alpha / \sqrt{N})^{2m} (\hat{S}_\beta / \sqrt{N})^{2\ell} \right\} = \frac{(2m)!(2\ell)!}{2^{3\ell} 2^{3m} \ell! m!}, \quad (19)$$

which means that

$$\lim_{N \rightarrow \infty} 2^{-N} \text{tr} \left\{ (\hat{S}_\alpha / \sqrt{N})^{2m} (\hat{S}_\beta / \sqrt{N})^{2\ell} \right\} = \lim_{N \rightarrow \infty} 2^{-N} \text{tr} (\hat{S}_\alpha / \sqrt{N})^{2m} \lim_{N \rightarrow \infty} 2^{-N} \text{tr} (\hat{S}_\beta / \sqrt{N})^{2\ell}. \quad (20)$$

Equation (14) determines the explicit form of the moments of the random variable, η , associated with the spectral decomposition of the operators \hat{S}_α/\sqrt{N} as N tends to infinity. To find the expression of the corresponding probability density function $p(\eta)$, we first note that the characteristic function is given in terms of the moments $\langle \eta^n \rangle$ by

$$\Phi(t) = \sum_{n=0}^{\infty} (it)^n \frac{\langle \eta^n \rangle}{n!}, \quad (21)$$

so that

$$\Phi(t) = \sum_{n=0}^{\infty} (it)^{2n} \frac{(2n)!}{2^{3n} n! (2n)!} = \sum_{n=0}^{\infty} \frac{(it)^{2n}}{2^{3n} n!}. \quad (22)$$

This yields

$$\Phi(t) = e^{-t^2/8}. \quad (23)$$

The probability density function is nothing but the Fourier transform of the characteristic function:

$$p(\eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(t) e^{-it\eta} dt. \quad (24)$$

By direct substitution of the expression of $\Phi(t)$ into the latter equation, we find that

$$p(\eta) = \sqrt{\frac{2}{\pi}} e^{-2\eta^2}, \quad (25)$$

that is, a Gaussian probability distribution with zero mean and variance $\sigma = \frac{1}{2}$.

A corollary of the above result is that for any well-behaved functions f, g :

$$\lim_{N \rightarrow \infty} 2^{-N} \text{tr} f\left(\frac{\hat{S}_\alpha}{\sqrt{N}}\right) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} f(\eta) e^{-2\eta^2} d\eta, \quad (26)$$

$$\lim_{N \rightarrow \infty} 2^{-N} \text{tr} g\left(\frac{\hat{S}_\alpha}{\sqrt{N}}, \frac{\hat{S}_\beta}{\sqrt{N}}\right) = \frac{2}{\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(\eta, \mu) e^{-2(\eta^2 + \mu^2)} d\eta d\mu, \quad \alpha, \beta \equiv x, y, z. \quad (27)$$

If, instead of the operators \hat{S}_x and \hat{S}_y , we use the lowering and raising operators \hat{S}_+ and \hat{S}_- , which are related to each other by

$$\hat{S}_x = \frac{1}{2}(\hat{S}_+ + \hat{S}_-), \quad \hat{S}_y = \frac{1}{2i}(\hat{S}_+ - \hat{S}_-), \quad (28)$$

the integral in equation (27) should be expressed in terms of the random variables z^* and z corresponding to \hat{S}_+ and \hat{S}_- , respectively. One can easily verify that

$$\lim_{N \rightarrow \infty} 2^{-N} \text{tr} g\left(\frac{\hat{S}_+}{\sqrt{N}}, \frac{\hat{S}_-}{\sqrt{N}}\right) = \frac{2}{\pi} \int_{z^*} \int_z g(z^*, z) e^{-2|z|^2} dz^* dz. \quad (29)$$

This being explicitly proved, we are ready now to formulate our main result in the next section.

III. BOSONIZATION

The crucial observation is that equation (29) is nothing but the coherent state representation of the mean value of the function operator g written in normal order (which we denote by $\mathcal{N}g$):

$$\frac{2}{\pi} \int_{z^*} \int_z g(z^*, z) e^{-2|z|^2} dz^* dz = \int_{z^*} \int_z P(z^*, z) \mathcal{N}g(z^*, z) dz^* dz. \quad (30)$$

Here $P(z^*, z)$ is the P-representation [13] of the density matrix $\hat{\rho}$ of an equivalent bosonic system (harmonic oscillator, as we shall see below) whose creation and annihilation operators are denoted by a^\dagger and a , respectively. Explicitly, we have

$$P(z^*, z) = \sum_n \langle n | \hat{\rho} \delta(z^* - a^\dagger) \delta(z - a) | n \rangle \quad (31)$$

which, obviously, is normalized to unity:

$$\int_{z^*} \int_z P(z^*, z) dz^* dz = 1. \quad (32)$$

It follows that the density matrix $\hat{\rho}$ is given by

$$\hat{\rho} = \int_{z^*} \int_z P(z^*, z) |z\rangle \langle z| dz^* dz, \quad (33)$$

with the coherent states

$$|z\rangle = e^{-|z|^2/2} \sum_{m=0}^{\infty} \frac{z^m}{m!} |m\rangle. \quad (34)$$

Whence

$$\langle n | \hat{\rho} | m \rangle = \frac{2}{\pi} \int_{z^*} dz^* \int_z dz e^{-3|z|^2} \frac{z^n z^{*m}}{\sqrt{n!m!}} \quad (35)$$

$$= \frac{2}{\pi} \int_0^{2\pi} d\phi \int_0^\infty e^{-3r^2} \frac{r^{n+m}}{\sqrt{n!m!}} e^{i(m-n)\phi} r dr \quad (36)$$

$$= \delta_{mn} \frac{2}{3^{n+1}}, \quad (37)$$

where use has been made of the polar coordinates (r, ϕ) . In fact we can rewrite the above result in the form

$$\langle n | \hat{\rho} | n \rangle = \frac{\left(\frac{1}{2}\right)^n}{\left(1 + \frac{1}{2}\right)^{n+1}}. \quad (38)$$

This reminds us the form of the occupation number representation of the density matrix of a harmonic oscillator in thermal equilibrium at temperature T , namely,

$$\hat{\rho} = \sum_n \frac{\langle n \rangle^n}{(1 + \langle n \rangle)^{n+1}} |n\rangle \langle n| \quad (39)$$

where the mean value is given in terms of the natural frequency of the oscillator by

$$\langle n \rangle = \text{tr}(\hat{\rho} a^\dagger a) = \frac{1}{e^{\frac{\hbar\omega}{k_B T}} - 1}. \quad (40)$$

Comparing equations (38) and (39), and using equation (40), we obtain

$$\langle n \rangle = \frac{1}{2}, \quad \frac{\hbar\omega}{k_B T} = \ln 3, \quad (41)$$

meaning that

$$\hat{\rho} = e^{-\ln 3(a^\dagger a + \frac{1}{2})}/Z, \quad Z = \frac{\sqrt{3}}{2}. \quad (42)$$

Our main result may thus be formulated as follows ($\hbar = 1$):

Theorem 1 *Given any well-behaved function f , we have:*

$$\lim_{N \rightarrow \infty} \text{tr} \left\{ \underbrace{\frac{\mathbb{I}_N}{2^N}}_{\hat{\rho}_s(T=\infty)} f\left(\frac{\hat{S}_+}{\sqrt{N}}, \frac{\hat{S}_-}{\sqrt{N}}\right) \right\} = \frac{2}{\sqrt{3}} \text{tr} \left\{ e^{-\ln 3(a^\dagger a + \frac{1}{2})} \mathcal{N} f(a^\dagger, a) \right\} \quad (43)$$

We conclude that at infinite temperature and large N , the raising and lowering scaled spin operators \hat{S}_+/\sqrt{N} and \hat{S}_-/\sqrt{N} behave like the creation and annihilation operators a^\dagger and a of a harmonic oscillator in thermal equilibrium whose frequency and temperature are related by $\hbar\omega/k_B T = \ln 3$.

Worked example

Let us illustrate the above result by considering the operator

$$f\left(\frac{\hat{S}_+}{\sqrt{N}}, \frac{\hat{S}_-}{\sqrt{N}}\right) = (\hat{S}_+ \hat{S}_- + \hat{S}_- \hat{S}_+)^5 / N^{5/2}. \quad (44)$$

For $N = 2000$ we find that

$$\text{tr} \left\{ \frac{\mathbb{I}_N}{2^N} f\left(\frac{\hat{S}_+}{\sqrt{N}}, \frac{\hat{S}_-}{\sqrt{N}}\right) \right\} = 119.670. \quad (45)$$

On the other hand,

$$\mathcal{N} f(a^\dagger, a) = 2^5 a^{\dagger 5} a^5 = 32[(a^\dagger a)^5 - 10(a^\dagger a)^4 + 35(a^\dagger a)^3 - 50(a^\dagger a)^2 + 24a^\dagger a]. \quad (46)$$

Taking into account the fact that

$$\frac{2}{\sqrt{3}} \sum_{n=0}^{\infty} e^{-\ln 3(n + \frac{1}{2})} n^k = \frac{2}{3} \text{Li}_{-k}\left(\frac{1}{3}\right) \quad (47)$$

where $\text{Li}_m(x)$ denotes the polylogarithmic function, we find that

$$\frac{2}{\sqrt{3}} \sum_{n=0}^{\infty} 32e^{-\ln 3(n+\frac{1}{2})} (n^5 - 10n^4 + 35n^3 - 50n^2 + 24n) = 120 \quad (48)$$

as should be.

What about the z component? To determine the equivalent degree of freedom, let us have a look at the probability density

$$x \mapsto p(x) = \sqrt{\frac{2}{\pi}} e^{-2x^2} \quad (49)$$

and compare it with the Gaussian probability distribution of a harmonic oscillator [see equation (26)]

$$x \mapsto |\psi(x)|^2 = \frac{1}{\sqrt{2\pi(\Delta x)^2}} \exp\left\{-\frac{(x - \langle x \rangle)^2}{2(\Delta x)^2}\right\}, \quad (50)$$

$$\langle f(x) \rangle = \int_{-\infty}^{+\infty} f(x) |\psi(x)|^2 dx. \quad (51)$$

We find that

$$\langle x \rangle = 0, \quad \Delta x = \frac{1}{2}. \quad (52)$$

For the ground state (that is a minimum uncertainty state), the variance is given in terms of the mass m and the frequency $\tilde{\omega}$ of the oscillator by $\Delta x = \sqrt{\frac{1}{2m\tilde{\omega}}}$ (remember that we set $\hbar = 1$). Hence

$$m\tilde{\omega} = 2. \quad (53)$$

This means that, as $N \rightarrow \infty$, the operator \hat{S}_z/\sqrt{N} behaves like the *position variable* of a harmonic oscillator (different from the above one) in its Gaussian ground state ($T = 0$) whose variance is equal to $1/2$. The full spin system is thus mapped onto a bipartite bosonic system, consisting of two independent harmonic oscillators, one of which is in thermal equilibrium, while the other one is at zero temperature.

Let us apply the above results to the Heisenberg XY Hamiltonian (γ is the coupling constant):

$$\hat{H} = \frac{\gamma}{N} (\hat{S}_+ \hat{S}_- + \hat{S}_- \hat{S}_+). \quad (54)$$

The thermal expectation value of the function operator $f(\hat{S}_+/\sqrt{N}, \hat{S}_-/\sqrt{N})$ is given by

$$\langle f \rangle = \frac{\text{tr } e^{-\frac{\gamma}{k_B T N} (\hat{S}_+ \hat{S}_- + \hat{S}_- \hat{S}_+)} f\left(\frac{\hat{S}_+}{\sqrt{N}}, \frac{\hat{S}_-}{\sqrt{N}}\right)}{\text{tr } e^{-\frac{\gamma}{k_B T N} (\hat{S}_+ \hat{S}_- + \hat{S}_- \hat{S}_+)}} \quad (55)$$

As $N \rightarrow \infty$, equation (43) implies that

$$\langle f \rangle = \frac{\text{tr } e^{-\ln 3(a^\dagger a + \frac{1}{2})} \mathcal{N} e^{-\frac{\gamma}{k_B T}(a^\dagger a + a a^\dagger)} f(a^\dagger, a)}{\text{tr } e^{-\ln 3(a^\dagger a + \frac{1}{2})} \mathcal{N} e^{-\frac{\gamma}{k_B T}(a^\dagger a + a a^\dagger)}}, \quad (56)$$

where γ is now given in units of \hbar . Taking into account the fact that

$$\mathcal{N}(a^\dagger a + a a^\dagger)^n = 2^n (a^\dagger)^n a^n = 2^n \sum_{\ell=0}^n B_\ell^n (a^\dagger a)^\ell \quad (57)$$

where B_ℓ^n denotes Stirling's numbers of the first kind [14], we find that

$$\mathcal{N} e^{-\frac{\gamma}{k_B T}(a^\dagger a + a a^\dagger)} = \sum_{n=0}^{\infty} \left(\frac{-2\gamma}{k_B T} \right)^n \frac{1}{n!} \sum_{\ell=0}^n B_\ell^n (a^\dagger a)^\ell. \quad (58)$$

By using the following formula for Stirling's numbers of the first kind:

$$(1+t)^u = \sum_{n=0}^{\infty} \sum_{k=0}^n B_k^n \frac{t^n}{n!} u^k, \quad (59)$$

we deduce that

$$\mathcal{N} e^{-\frac{\gamma}{k_B T}(a^\dagger a + a a^\dagger)} = (1 - 2\gamma/(k_B T))^{a^\dagger a} = e^{\ln(1 - \frac{2\gamma}{k_B T}) a^\dagger a}. \quad (60)$$

From equation (56), we find that the partition function of the corresponding bosonic system is given by

$$Z = \text{tr} \left\{ e^{-\ln \left(\frac{3}{1 - 2\gamma/k_B T} \right) (a^\dagger a + \frac{1}{2})} \right\}, \quad (61)$$

which is the partition function for a single mode harmonic oscillator in thermal equilibrium.

The following conditions should however be satisfied:

$$1 > \frac{2\gamma}{k_B T}, \quad 1 > -\frac{\gamma}{k_B T}. \quad (62)$$

In the case of ferromagnetic interactions, $\gamma < 0$, we find that

$$k_B T > |\gamma|. \quad (63)$$

This gives the lower bound on the temperature above which the bosonization is valid. To proceed further, let us apply equation (7) to the low excitation sector of the Hamiltonian; one finds

$$\hat{H} = \gamma(\alpha_0^\dagger \alpha_0 + h.c) = 2\gamma(\alpha_0^\dagger \alpha_0 + \frac{1}{2}). \quad (64)$$

Hence,

$$k_B T > \frac{\hbar \omega_0}{2}, \quad \hbar \omega_0 = 2|\gamma|. \quad (65)$$

We may thus define an effective temperature for the harmonic oscillator as follows:

$$k_B T_{\text{eff}} = \frac{2|\gamma|}{\ln\left(\frac{3}{1-2\gamma/k_B T}\right)}. \quad (66)$$

Finally, regarding the function f , it can be mapped according to

$$f\left(\frac{\hat{S}_+}{\sqrt{N}}, \frac{\hat{S}_-}{\sqrt{N}}\right) \mapsto e^{-\ln(1-2\gamma/k_B T)a^\dagger a} \mathcal{N} e^{-\frac{2\gamma}{k_B T}(a^\dagger a + \frac{1}{2})} f(a^\dagger, a). \quad (67)$$

IV. DISCUSSION AND CONCLUDING REMARKS

In conclusion we have established a bosonic representation for scaled total spin operators at high temperatures. It should be pointed out that this bosonization scheme does not preserve the temperature, since the spin system, originally taken at infinite temperature is mapped onto a bosonic system at finite temperature. Equation (41) does not determine the exact values of the parameters ω and T , but knowing the value of the fraction is sufficient to fully determine the trace. This can be physically explained by the fact that the original spin system is fully randomized at infinite temperature; the number of degrees of freedom necessary to fully characterize it is equal to the number of the generators \hat{S}_k , namely, 3. But for the harmonic oscillator, we have only two generators; the z component is found to be equivalent to the position variable of another harmonic oscillator that is at zero temperature. This discrepancy is removed by the value $\ln 3$ of the fraction $\hbar\omega/k_B T$ which is clearly the entropy of a fully mixed three-level system. We have applied the formalism to the Heisenberg XY model at finite temperature, and found that it is equivalent to a single mode harmonic oscillator in thermal equilibrium which can be assigned an effective temperature that depends logarithmically on the physical one. Moreover, we have determined the exact rule for mapping any function of the spin operators to the bosonic counter part. The presented results in this paper are a first attempt and further investigation needs to be done to gain more insight into other possible applications.

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